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Pascal's Triangle and Mathematical Induction

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1 A Review of the Figurate Numbers

Recall that the figurate numbers count the number of dots in regularly shaped figures, such as line segments, triangles, pyramids, etc. A line segment is a one-dimensional object, a triangle is a two-dimensional object, while a pyramid is a three-dimensional object. There are surprisingly four-dimensional figurate numbers and in fact figurate numbers in any dimension beyond this. Is there any unifying principle for the construction of these numbers or must each class of figurate numbers be studied separately in its own dimension? Let's start with what Nicomachus of Gerasa (first century BCE) calls "the beginning of dimension," [4, Ch. VII] [5], namely a "point" with "unity [...] occupying the place and character of a point." We now compare the following statements of Nicomachus [4, 5] and Pascal [7, p. 455]

Nicomachus: The writing of one unit [...] will be the sign for 1.

Pascal: I call *numbers of the first order* the units,

$$1,\ 1,\ 1,\ 1,\ 1,\ \text{etc.}$$

Nicomachus: [L]inear numbers [...] advance by the addition of 1 [...].

Pascal: I call *numbers of the second order* the natural numbers which are formed by the addition of units,

$$1,\ 2,\ 3,\ 4,\ 5,\ \text{etc.}$$

If we introduce separate symbols $L_1,\ L_2,\ L_3,\ \ldots$ for the first, second, third, ... linear numbers, then

$$L_1 = 1, \quad L_2 = 2, \quad L_3 = 3,\ \ldots\ .$$

Note that $L_2 = L_1 + 1$, $L_3 = L_2 + 1$, ...

Exercise 1.1. Let L_4 denote the fourth linear number. Find an equation relating L_4 to L_3 so that

$$L_4 = L_3 + \boxed{}.$$

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Exercise 1.2. For a generic natural number n, let L_n denote the nth linear number. Find an equation relating L_n to the preceding linear number L_{n-1} so that

$$L_n = L_{n-1} + \boxed{}.$$

This is the recursion formula for the linear numbers.

Exercise 1.3. In what dimension are the figurate numbers that Pascal refers to as "numbers of the second order"? Is Pascal's use of the word "order" the same as our use of the word "dimension"?

Nicomachus: The triangular number is produced from the natural series of number [...] by the continued addition of successive terms.

Pascal: I call *numbers of the third order*, known as triangular numbers, those which are formed by the addition of the natural numbers,

1, 3, 6, 10, etc.

Let's introduce separate symbols T_1, T_2, T_3, ... for the first, second, third, ... triangular numbers. Then $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, etc. Note that

$$\begin{aligned} T_2 &= 1+2, & T_2 &= L_1 + L_2 = T_1 + L_2, \\ T_3 &= 1+2+3, & T_3 &= (L_1 + L_2) + L_3 = T_2 + L_3. \end{aligned}$$

Exercise 1.4. Let T_4 denote the fourth triangular number.

(a) Compute the numerical value for T_4. Be sure to explain the strategy for your calculation.

(b) Write T_4 as the sum of four consecutive natural numbers so that

$$T_4 = \boxed{} + \boxed{} + \boxed{} + \boxed{}.$$

(c) Fill in the blank with a natural number so that

$$T_4 = T_3 + \boxed{}.$$

(d) Write the equation in part (c) in terms of an appropriate linear number so that

$$T_4 = T_3 + L_{\boxed{}}.$$

Exercise 1.5. For a generic natural number n, let T_n denote the nth triangular number. Let's find an equation relating T_n to the preceding triangular number T_{n-1}.

(a) Fill in the blank with a natural number so that

$$T_n = T_{n-1} + \boxed{}.$$

(b) Write the equation in part (a) in terms of an appropriate linear number so that

$$T_n = T_{n-1} + L_{\square}.$$

This is known as the recursion formula for the triangular numbers.

Exercise 1.6. In what dimension are the figurate numbers that Pascal refers to as "numbers of the third order"? Is Pascal's use of the word "order" the same as our use of the word "dimension"?

Nicomachus: The pyramids with a triangular base, then, in their proper order are these: 1, 4, 10, 20, 35, 56, 84, and so on; and their origin is the piling up of the triangular numbers one upon the other.

Pascal: I call *numbers of the fourth order* those which are formed by addition of the triangular numbers,

1, 4, 10, 20, etc.

Now let's introduce separate symbols for the pyramidal numbers. Let $P_1, P_2, P_3, \ldots$ denote the first, second, third, ... pyramidal numbers. Thus, $P_1 = 1$, $P_2 = 4$, $P_3 = 10$, etc. Note that

$$\begin{aligned} P_2 &= 1 + 3, & P_2 &= T_1 + T_2 = P_1 + T_2, \\ P_3 &= 1 + 3 + 6, & P_3 &= (T_1 + T_2) + T_3 = P_2 + T_3. \end{aligned}$$

Exercise 1.7. Let P_4 denote the fourth pyramidal number.

(a) Compute the numerical value for P_4. Be sure to explain the strategy for your calculation.

(b) Write P_4 as the sum of four natural numbers so that

$$P_4 = \square + \square + \square + \square.$$

(c) Fill in the blank with a natural number so that

$$P_4 = P_3 + \square.$$

(d) Write the equation in part (c) in terms of an appropriate triangular number so that

$$P_4 = P_3 + T_{\square}.$$

Exercise 1.8. For a generic natural number n, let P_n denote the nth pyramidal number. Find an equation relating P_n to the preceding pyramidal number P_{n-1} in terms of an appropriate triangular number so that

$$P_n = P_{n-1} + T_{\square}.$$

This is known as the recursion relation for the pyramidal numbers.

Exercise 1.9. In what dimension are the figurate numbers that Pascal refers to as "numbers of the fourth order"? Is Pascal's use of the word "order" the same as our use of the word "dimension"?

For the ancient Greeks, objects did not exist beyond the third dimension. After listing the first few pyramidal numbers, Pascal [7, p. 455] continues:

I call *numbers of the fifth order* those which are formed by addition of the preceding numbers. Since there is no fixed name for them, they might be called triangulo-triangular numbers:

$$1,\ 5,\ 15,\ 35, \text{etc.}$$

Let's denote the first, second, third, ... triangulo-triangular numbers by $Q_1, Q_2, Q_3, \ldots$. Then, $Q_1 = 1$, $Q_2 = 5$, $Q_3 = 15$, etc. Note that

$$\begin{aligned} Q_2 &= 1 + 4, & Q_2 &= P_1 + P_2 = Q_1 + P_2, \\ Q_3 &= 1 + 4 + 10, & Q_3 &= (P_1 + P_2) + P_3 = Q_2 + P_3. \end{aligned}$$

Exercise 1.10. Let Q_4 denote the fourth triangulo-triangular number.

(a) Compute the numerical value for Q_4. Be sure to explain the strategy for your calculation.

(b) Write Q_4 as the sum of four consecutive pyramidal numbers so that

$$Q_4 = P_{\square} + P_{\square} + P_{\square} + P_{\square}.$$

(c) Find an appropriate pyramidal number so that

$$Q_4 = Q_3 + P_{\square}.$$

Exercise 1.11. For a generic natural number n, let Q_n denote the nth triangulo-triangular number. Find an equation relating Q_n to the preceding triangulo-triangular number Q_{n-1} in terms of an appropriate pyramidal number so that

$$Q_n = Q_{n-1} + P_{\square}.$$

This is known as the recursion relation for the triangulo-triangular numbers.

Exercise 1.12. In what dimension are the triangulo-triangular numbers?

After introducing the triangulo-triangular numbers, Pascal writes:

I call *numbers of the sixth order* those which are formed by the addition of the preceding numbers,

1, 6, 21, 56, 126, 252, etc.

And so to infinity, 1, 7, 28, 84, etc.

1, 8, 36, 120, etc.

Exercise 1.13. Explain how the figurate numbers of the sixth order, 1, 6, 21, 56, 126, ... are computed from the triangulo-triangular numbers. You may introduce special notation for the figurate numbers of the sixth order, such as R_1, R_2, R_3, ... for the first, second, third, ... of these numbers.

Exercise 1.14. For a generic natural number n, let R_n denote the nth figurate number of order six. Develop a recursion relation for R_n that expresses how R_n is computed from the previous figurate number of order six, R_{n-1}, and a certain triangulo-triangular number. Be sure to explain your work.

Exercise 1.15. Is there a recursion relation for the figurate numbers of the first order? Be sure to explain your work. In what dimension are these numbers of the first order? Hint: In what dimension is a point?

Exercise 1.16. Extra. Based on the work in this section, find a recursion relation for $E_{k,n}$, the nth figurate number of order k. Be sure to explain your work. In what dimension is a figurate number of order k?

2 Pascal's Arithmetical Triangle

Now that we have reviewed the figurate numbers in various dimensions (or in various orders), what would result if a table were constructed that contained all of these numbers? Would a jumble of numbers result or would there be patterns showing how the figurate numbers are related to each other? This may depend on how the table is constructed. Suppose that the figurate numbers of order k are placed one by one in row k of the table. First of all, is there a simple organizing principle that guides the construction of such a table? We have seen that there are recursive formulas for each order of the figurate numbers. Is there one all-inclusive recursive formula for these numbers of all orders? To answer this, let's read in English translation [7] a few excerpts from Pascal's original paper *Traité du triangle arithmétique* [6], published posthumously in 1665.

Blaise Pascal, from

TREATISE ON THE ARITHMETICAL TRIANGLE

Definitions

I call *arithmetical triangle* a figure constructed as follows:

From any point, G, I draw two lines perpendicular to each other, GV, Gζ in each of which I take as many equal and contiguous parts as I please, beginning with G, which I number 1, 2, 3, 4, etc., and these numbers are the *exponents* of the sections of the lines.

Next I connect the points of the first section in each of the two lines by another line, which is the base of the resulting triangle.

In the same way I connect the two points of the second section by another line, making a second triangle of which it is the base.

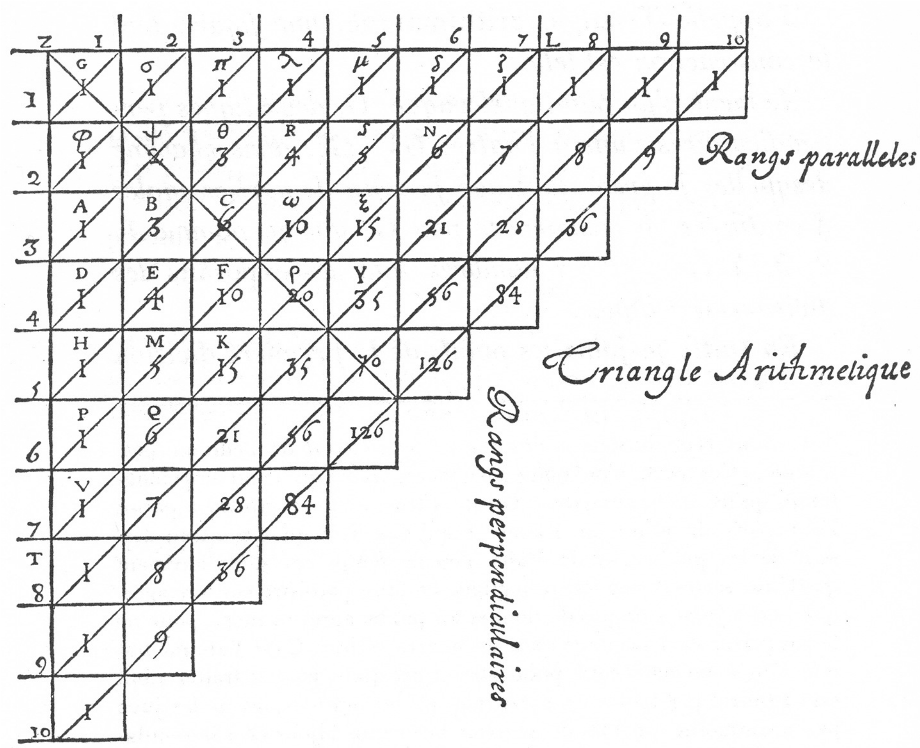

Figure 1: Pascal's Arithmetical Triangle.

And in this way connecting all the points of section with the same exponent, I construct as many triangles and bases as there are exponents.

Through each of the points of section and parallel to the sides I draw lines whose intersections make little squares which I call *cells*.

Cells between two parallels drawn from left to right are called *cells of the same parallel row*, as, for example, cells G, σ, π, etc., or φ, ψ, θ, etc.

Those between two lines are drawn from top to bottom are called *cells of the same perpendicular row*, as, for example, cells G, φ, A, D, etc., or σ, ψ, B, etc.

Those cut diagonally by the same base are called *cells of the same base*, as, for example, D, B, θ, λ, or A, ψ, π.

Cells of the same base equidistant from its extremities are called *reciprocals*, as, for example, E, R and B, θ, because the parallel exponent of one is the same as the perpendicular exponent of the other, as is apparent in the above example, where E is in the second perpendicular row and in the fourth parallel row and its reciprocal, R, is in the second parallel row and in the fourth perpendicular row, reciprocally. It is very easy to demonstrate that cells with exponents reciprocally the same are in the same base and are equidistant from its extremities.

It is also very easy to demonstrate that the perpendicular exponent of any cell when added to is parallel exponent exceeds by unity the exponent of its base.

For example, cell F is in the third perpendicular row and in the fourth parallel row and in the sixth base, and the exponents of rows 3 and 4, added together, exceed by unity the exponent of base 6, a property which follows from the fact that the two sides of the triangle have the same number of parts; but this is understood rather than demonstrated.

Of the same kind is the observation that each base has one more cell than the preceding base, and that each has as many cells as its exponent has units; thus the second base, $\varphi\sigma$, has two cells, the third, $A\psi\pi$, has three, etc.

Now the numbers assigned to each cell are found by the following method:

The number of the first cell, which is at the right angle, is arbitrary; but that number having been assigned, all the rest are determined, and for this reason it is called the *generator* of the triangle. Each of the others is specified by a single rule as follows:

The number of each cell is equal to the sum of the numbers of the perpendicular and parallel cells immediately preceding. Thus cell F, that is, the number of cell F, equals the sum of cell C and cell E, and similarly with the rest.

Whence several consequences are drawn. The most important follow, wherein I consider triangles generated by unity, but what is said of them will hold for all others.

Exercise 2.1. Pascal begins his table (arithmetical triangle) by drawing two lines GV and $G\zeta$ perpendicular to each other. He then sections each of these lines into equal parts. Explain what is meant by an "exponent" of the sections of these lines.

Exercise 2.2. What is meant by the "base" of a triangle in Pascal's table? List the letters of the cells in the base of the second triangle in Pascal's table.

Exercise 2.3. List the letters in the base of the third triangle in Pascal's table.

Exercise 2.4. What is meant by the term "cells of the same parallel row"?

Exercise 2.5. What is meant by the term "cells of the same perpendicular row"? Would the term "column" be more appropriate today to describe a "perpendicular row"? Why, in your opinion, does Pascal not use the term "column"?

Exercise 2.6. Carefully read again how Pascal defines the "reciprocal" of a cell.

(a) What is meant by the term "parallel exponent" of a cell? Would the term "row number" of a cell be more appropriate? Why or why not?

(b) What is meant by the term "perpendicular exponent" of a cell? Would the term "column number" of a cell be more appropriate? Why or why not?

(c) Explain exactly what is meant by the "reciprocal" of a cell.

(d) Find the row and column number of the cell labeled by the letter M in Pascal's table.

(e) What is the reciprocal of cell M? Explain your answer.

(f) Find the row and column number of the cell labeled by the letter K in Pascal's table.

(g) What is the reciprocal of cell K? Be sure to explain your answer.

(h) Find the row and column number of the cell labeled by the letter ρ in Pascal's table.

(i) What is the reciprocal of cell ρ? Be sure to explain your answer.

Exercise 2.7. Let's identify the figurate numbers that appear in Pascal's table (arithmetical triangle).

(a) What figurate numbers appear in the first row of Pascal's table? What is their order (see the previous section)? What is their dimension?

(b) What figurate numbers appear in the second row of Pascal's table? What is their order? What is their dimension?

(c) What figurate numbers appear in the third row of Pascal's table? What is their order? What is their dimension?

(d) What figurate numbers appear in the fourth row of Pascal's table? What is their order? What is their dimension?

(e) What figurate numbers appear in the fifth row of Pascal's table? What is their order? What is their dimension?

Exercise 2.8. What cell is the "generator" of Pascal's table? What is the numerical value of this generating cell?

Exercise 2.9. Pascal claims that once the value of the generating cell is given, then the value of every other cell in the table is determined by a single rule: "The number of each cell is equal to the sum of the numbers of the perpendicular and parallel cells immediately preceding." We will call this the construction principle for Pascal's table.

(a) Explain how the numbers in the first row of Pascal's table can be computed using the construction principle. You may assume that any cell above the first row has value zero.

(b) Explain how the numbers in the second row of Pascal's table can be computed using the construction principle. In this row, how does the construction principle become the recursion formula for the linear numbers?

(c) Explain how the numbers in the third row of Pascal's table can be computed using the construction principle. In this row, how does the construction principle become the recursion formula for the triangular numbers?

(d) Explain how the numbers in the fourth row of Pascal's table can be computed using the construction principle. In this row, how does the construction principle become the recursion formula for the pyramidal numbers?

(e) Explain how the numbers in the fifth row of Pascal's table can be computed using the construction principle. In this row, how does the construction principle become the recursion formula for the triangulo-triangular numbers?

(f) Explain how the numbers in the sixth row of Pascal's table can be computed using the construction principle. In this row, how does the construction principle become the recursion formula for the figurate numbers of order six?

(g) Has Pascal found a universal recursion formula for generating figurate numbers of any order? Why or why not?

3 Pascal's First Consequence

The first row of Pascal's table (arithmetical triangle) begins with the units: 1, 1, 1, ... If we were to guess the next entry in this row, what would it be? What would the justification be other than a pattern of three "1"s is followed by a fourth "1"? Pascal finds many patterns in his table and carefully justifies each one with an increasing level of rigor as he proceeds. His main tool is the construction principle, which he states verbally as "The number of each cell is equal to the sum of the numbers of the perpendicular and parallel cells immediately preceding." Many patterns follow from this principle and these patterns are "consequences" of the construction principle. Additionally Pascal claims that these patterns hold no matter how far the table is continued. What type of reasoning would allow us to conclude that a pattern continues infinitely far? Let's begin to answer this as we read from Pascal's treatise [7].

First Consequence

In every arithmetical triangle all the cells of the first parallel row and of the first perpendicular row are the same as the generating cell.

For by definition each cell of the triangle is equal to the sum of the immediately preceding perpendicular and parallel cells. But the cells of the first parallel row have no preceding perpendicular cells, and those of the first perpendicular row have no preceding parallel cells; therefore they are all equal to each other and consequently to the generating number.

Thus $\varphi = G + 0$, that is, $\varphi = G$,

$A = \varphi + 0$, that is, φ,

$\sigma = G + 0$, $\pi = \sigma + 0$,

And similarly of the rest.

Exercise 3.1. Locate the generating cell in Pascal's table (arithmetical triangle).

(a) What letter does Pascal use to denote the generating cell?

(b) What is the numerical value of the generating cell?

Exercise 3.2. For each of the following steps provide a reason for its validity. The reason may be as simple as "the construction principle" or some rule of algebra.

Statement	Reason
1. $\sigma = G + 0$	1.
2. $\sigma = G$	2.
3. $\pi = \sigma + 0$	3.
4. $\pi = \sigma$	4.
5. $\pi = G$	5.

Pascal concludes his argument with "And similarly of the rest." Do you feel that Pascal has justified why all cells of the first row are the same as the generating cell? Why or why not?

Exercise 3.3. Reread the five steps in Exercise (3.2) above.

(a) Continue the argument to show that $\lambda = G$. Be sure to offer a reason why each step holds.

(b) Continue the argument to show that $\mu = G$. Be sure to offer a reason why each step holds.

(c) Explain how this argument would apply to any cell of the first row. Are there any logical difficulties that you encounter? If so, what are they?

4 Pascal's Second Consequence

Recall from section one that both Nicomachus and Pascal state that the triangular numbers can be produced by the addition of successive natural numbers. This leads to what today would be called the iterative formula for the triangular numbers. For a generic natural number n, let T_n denote the nth triangular number. This iterative formula becomes:

$$T_n = 1 + 2 + 3 + \ldots + n.$$

If we let L_1, L_2, L_3, … denote the first, second, third, … linear numbers, then this iterative formula can be expressed entirely in symbolic form as:

$$T_n = L_1 + L_2 + L_3 + \ldots + L_n.$$

Similarly, both Nicomachus and Pascal state that a pyramidal number can be produced by the addition (or piling up) of successive triangular numbers. This leads to the iterative formula for P_n, the nth pyramidal number. We have:

$$P_n = T_1 + T_2 + T_3 + \ldots + T_n.$$

Likewise there are iterative formulas for the triangulo-triangular numbers, the figurate numbers of order six, the figurate numbers of order seven, etc.

Exercise 4.1. For a generic natural number n, let Q_n denote the nth triangulo-triangular number. Find a formula for Q_n that expresses Q_n as the sum of the first n-many pyramidal numbers. This is known today as the iterative formula for the triangulo-triangular numbers.

Is there a method of expressing each of these iterative formulas as an all-inclusive iterative construction? If so, what would it be and would it follow from Pascal's simple rule that we have called the construction principle?

SECOND CONSEQUENCE

In every arithmetical triangle each cell is equal to the sum of all the cells of the preceding parallel row from its own perpendicular row to the first, inclusive.

Let any cell, ω, be taken. I say that it is equal to $R + \theta + \psi + \varphi$, which are the cells of the next higher parallel row from the perpendicular row of ω to the first perpendicular row.

This is evident if we simply consider a cell as the sum of its component cells.

For ω equals $R + \underbrace{C}_{\theta + \underbrace{B}_{\psi + \underbrace{A}_{\varphi,}}}$

for A and φ are equal to each other by the preceding consequence.

Therefore $\omega = R + \theta + \psi + \varphi$.

Exercise 4.2. Locate the cell labeled ω in Pascal's table (arithmetical triangle).

(a) What is the row number of ω?

(b) What is the column number of ω?

(c) What triangular number is represented by ω? Fill in the blank with a specific natural number so that

$$\omega = T_{\square}.$$

Exercise 4.3. Provide a simple reason to justify each of the following steps showing that ω "is equal to the sum of all the cells of the preceding row from its own perpendicular row to the first, inclusive." The reasons may be such things as "the construction principle," "Pascal's First Consequence," or a rule of algebra.

Statement	Reason
1. $\omega = R + C$	1.
2. $C = \theta + B$	2.
3. $\omega = R + \theta + B$	3.
4. $B = \psi + A$	4.
5. $\omega = R + \theta + \psi + A$	5.
6. $A = \varphi$	6.
7. $\omega = R + \theta + \psi + \varphi$	7.

Exercise 4.4. What linear numbers are represented by R, θ, ψ, φ? Find specific natural numbers so that

$$R = L_{\square}, \quad \theta = L_{\square}, \quad \psi = L_{\square}, \quad \varphi = L_{\square}.$$

Exercise 4.5. Carefully explain how the equation

$$\omega = R + \theta + \psi + \varphi$$

is the iterative formula for the triangular number T_4.

Exercise 4.6. Using the result of Exercise (4.3), show that

$$\xi = S + R + \theta + \psi + \varphi.$$

Organize your work in a step-by-step argument and provide a reason why each step holds. You may only need two or three steps.

Exercise 4.7. We examine how the equation for ξ is another example of an iterative formula for a triangular number.

(a) What triangular number is represented by ξ?

(b) What linear numbers are represented by S, R, θ, ψ, φ?

(c) Explain how

$$\xi = S + R + \theta + \psi + \varphi$$

is the iterative formula for T_5.

Exercise 4.8. Carefully explain how the iterative formula for T_6 can be derived from the iterative formula for T_5 by using the construction principle.

Exercise 4.9. Is the construction principle enough to imply the validity of the iterative formula for T_n, where n is a generic natural number? Why or why not?

Exercise 4.10. In a step-by-step argument, show that

$$\rho = \omega + C + B + A.$$

Be sure to provide a reason why each step holds.

Exercise 4.11. Carefully explain how the equation

$$\rho = \omega + C + B + A$$

is an example of an iterative formula for a pyramidal number. Begin by identifying what pyramidal number is represented by ρ. Then identify the triangular numbers represented by ω, C, B, A.

5 Pascal's Fifth Consequence

Pascal describes the numbers in the first row of his table as numbers of the first order [7, p. 455], which are simply the units. He describes the numbers in the second row as numbers of the second order, which are just the linear numbers. Similarly the numbers in the third row are called numbers of the third order, which are the triangular numbers. Pascal has already demonstrated that the numbers in the first column of his table match those in the first row (The First Consequence). The reader may have noticed that the linear numbers also appear in the second column of Pascal's table and the triangular numbers appear in the third column. How did this happen? Are the numbers in column k of Pascal's table the same as the numbers in row k? What type of construction would switch a row number and a column number of a cell? Pascal has already drawn attention to this in his idea of the reciprocal of a cell. If a pattern is found in the reciprocal cells, would this pattern continue no matter how far the table is constructed? Let's read Pascal's reasoning process about this [7].

Fifth Consequence

In every arithmetical triangle each cell is equal to its reciprocal.

For in the second base, $\varphi\sigma$, it is evident that the two reciprocal cells, φ, σ, are equal to each other and to G.

In the third base, A, ψ, π, it is also obvious that the reciprocals, π, A, are equal to each other and to G.

In the fourth base it is obvious that the extremes, D, λ, are again equal to each other and to G.

And those between, B, θ, are obviously equal since $B = A + \psi$ and $\theta = \pi + \psi$. But $\pi + \psi = A + \psi$ by what has just been shown. Therefore, etc.

Similarly it can be shown for all the other bases that reciprocals are equal, because the extremes are always equal to G and the rest can always be considered as the sum of cells in the preceding base which are themselves reciprocals.

Exercise 5.1. Locate the first base in Pascal's table.

(a) How many cells are contained in the first base?

(b) What is the reciprocal of the cell in the first base?

(c) Is the cell in the first base equal to its reciprocal?

Exercise 5.2. Locate the second base in Pascal's table.

(a) What is the row number of φ?

(b) What is the column number of φ?

(c) What is the row number of σ?

(d) What is the column number of σ?

(e) Are and φ and σ reciprocal cells? Why or why not?

(f) As numbers, is $\varphi = \sigma$? Why or why not?

(g) Does Pascal's Fifth Consequence hold in the second base? Why or why not?

Exercise 5.3. Locate the third base of Pascal's table.

(a) What cell is the reciprocal of cell A?

(b) What cell is the reciprocal of cell ψ?

(c) What cell is the reciprocal of cell π?

(d) Are the reciprocal cells in the third base equal? Why or why not?

Exercise 5.4. Pascal develops a different reasoning process when showing equality of reciprocals in the next (fourth) base.

(a) List the cells in the fourth base.

(b) What cells in the fourth base are contained in either the first row or the first column? Why are these two cells equal?

(c) Provide a reason why each step in the following argument holds. The reason may be "the construction principle," "equality of reciprocals in the third base," or a known rule of algebra.

Statement	Reason
1. $B = A + \psi$	1.
2. $\theta = \pi + \psi$	2.
3. $A = \pi$	3.
4. $A + \psi = \pi + \psi$	4.
5. $B = \theta$	5.

(d) Does Pascal's Fifth Consequence hold in the fourth base? Why or why not?

Exercise 5.5. Locate the fifth base in Pascal's table.

(a) What is the reciprocal of cell E?

(b) Write E as the sum of two other cells by applying the construction principle.

(c) Write R as the sum of two other cells by applying the construction principle.

(d) In a step-by-step argument, show that $E = R$ by applying parts (b), (c) and then using the equality of reciprocal cells in the previous (fourth) base. Offer a reason why each step holds.

Exercise 5.6. Pascal claims that the Fifth Consequence holds in all other bases of his table, "because the extremes are always equal to G and the rest can always be considered as the sum of cells in the preceding base which are themselves reciprocals." Can you explain Pascal's reasoning process? Do you feel that this justifies why the Fifth Consequence holds in all bases? Why or why not?

6 Pascal's Twelfth Consequence

When Pascal reaches the Twelfth Consequence of his *Treatise on the Arithmetical Triangle* he has identified patterns in his table that are subtle enough to determine the value of any cell via a simple calculational procedure. Moreover, he develops a reasoning process to ensure that these patterns continue without fail no matter how far the table is continued. Of course, some of Pascal's patterns build on the work of Pierre de Fermat (1601–1665) and others. Recall that in a letter to Gilles Persone de Roberval dated November 4, 1636, Fermat [2, p. 83–87] [3, p. 230] claims of the triangular numbers that "The last side multiplied by the next larger makes twice the triangle," meaning

$$n(n+1) = 2T_n.$$

In the same letter Fermat writes of the pyramidal numbers that "The last side multiplied by the triangle of the next larger makes three times the pyramid," meaning

$$n \cdot T_{n+1} = 3P_n.$$

Of the triangulo-triangular numbers, he writes "The last side multiplied by the pyramid of the next greater makes four times the triangulo-triangle," meaning

$$n \cdot P_{n+1} = 4Q_n.$$

Fermat claims that these types of relations between figurate numbers continue to infinity, meaning in all dimensions (or orders), but he offers no verification. In his treatise Pascal [7, p. 472] writes "A number of whatever order when multiplied by the preceding root is equal to the exponent of its order multiplied by the preceding number of the following order." Although this statement is lengthy, let's apply it to T_{n+1}, a figurate number of order three and root $n+1$ (the sequence number in listing the triangles). The preceding root is n and $n \cdot T_{n+1}$ "is equal to the exponent of its order [3] multiplied by the preceding number of the following order [P_n]." In symbols, $n \cdot T_{n+1} = 3P_n$. Of Pascal's statement about "A number of whatever order," he writes [7, p. 473] "This very proposition ... occurred to our celebrated councilor of Toulouse, M. de Fermat, and the wonderful thing is that without either of us having given the other any slightest hint of what we were doing, he was writing in his province what I was discovering in Paris, and at the very hour, as our letters written and received at the same time bear witness."

To gain insight into Pascal's statement, it may be more efficient to use the row number of a cell as its "order" and the column number of a cell as its "root." Let's now introduce the notation $E_{k,n}$ for the cell in Pascal's table in row k and column n.

Exercise 6.1. Let $E_{k,n}$ denote the cell in row k and column n of Pascal's table. List the numerical value of the following cells (all of which are in the tenth base of the table).

$E_{10,1} =$	$E_{9,2} =$	$E_{8,3} =$	$E_{7,4} =$
$E_{6,5} =$	$E_{5,6} =$	$E_{4,7} =$	$E_{3,8} =$
$E_{2,9} =$	$E_{1,10} =$		

Exercise 6.2. Compute the following ratios of adjacent cells in the tenth base of Pascal's table. Can you identify a pattern?

$$\frac{E_{9,2}}{E_{10,1}} = \qquad \frac{E_{8,3}}{E_{9,2}} = \qquad \frac{E_{7,4}}{E_{8,3}} =$$
$$\frac{E_{6,5}}{E_{7,4}} = \qquad \frac{E_{5,6}}{E_{6,5}} = \qquad \frac{E_{4,7}}{E_{5,6}} =$$
$$\frac{E_{3,8}}{E_{4,7}} = \qquad \frac{E_{2,9}}{E_{3,8}} = \qquad \frac{E_{1,10}}{E_{2,9}} =$$

Exercise 6.3. Let $E_{k,n}$ be a cell in the tenth base of Pascal's table as in Exercise (6.2).

(a) Did you find a pattern for the ratios

$$\frac{E_{k-1,n+1}}{E_{k,n}} \;?$$

If so, state what the pattern is. If not, what difficulty did you have in finding a pattern?

(b) Fill in the blank so that the pattern above has form

$$\frac{E_{k-1,n+1}}{E_{k,n}} = \frac{\boxed{}}{n}.$$

(c) Write this pattern in the form

$$n \cdot E_{k-1,n+1} = \boxed{}$$

and verbally state this equation in words.

(d) Does your verbal description in part (c) have the same meaning as "A number of whatever order when multiplied by the preceding root is equal to the exponent of its order multiplied by the preceding number of the following order"? Why or why not?

Exercise 6.4. To prepare us for Pascal's Twelfth Consequence, let's return to the ratios of consecutive cells in the tenth base, Exercise (6.2).

(a) State again the value of the ratio

$$\frac{E_{7,4}}{E_{8,3}}.$$

(b) Counting cells diagonally along the tenth base, how far is $E_{7,4}$ from the top of the tenth base, where $E_{7,4}$ and $E_{1,10}$ are included in the count?

(c) Counting cells diagonally along the tenth base, how far is $E_{8,3}$ from the bottom of the tenth base, where $E_{8,3}$ and $E_{10,1}$ are included in the count?

(d) State again the value of the ratio

$$\frac{E_{2,9}}{E_{3,8}}.$$

(e) Counting cells diagonally along the tenth base, how far is $E_{2,9}$ from the top of the tenth base, where $E_{2,9}$ and $E_{1,10}$ are included in the count?

(f) Counting cells diagonally along the tenth base, how far is $E_{3,8}$ from the bottom of the tenth base, where $E_{3,8}$ and $E_{10,1}$ are included in the count?

In his Twelfth Consequence, Pascal not only identifies a pattern in the ratios of consecutive entries in the same base of his table, but he also develops a method of reasoning to ensure that this pattern continues to any base of the table.

Twelfth Consequence

In every arithmetical triangle, of two contiguous cells in the same base the upper is to the lower as the number of cells from the upper to the top of the base is to the number of cells from the lower to the bottom of the base, inclusive.

Let any two contiguous cells of the same base, E, C, be taken. I say that

E	:	C	::	2	:	3
the lower		the upper		because there are two cells from E to the bottom, namely E, H,		because there are three cells from C to the top, namely C, R, μ.

Although this proposition has an infinity of cases, I shall demonstrate it very briefly by supposing two lemmas:

The first, which is self-evident, that this proportion is found in the second base, for it is perfectly obvious that $\varphi : \sigma :: 1 : 1$;

The second, that if this proportion is found in any base, it will necessarily be found in the following base.

Whence it is apparent that it is necessarily in all the bases. For it is in the second base by the first lemma; therefore by the second lemma it is in the third base, therefore in the fourth, and to infinity.

It is only necessary therefore to demonstrate the second lemma as follows: If this proportion is found in any base, as, for example, in the fourth, $D\lambda$, that is, if $D : B :: 1 : 3$, and $B : \theta :: 2 : 2$, and $\theta : \lambda :: 3 : 1$, etc., I say the same proportion will be found in the following base, $H\mu$, and that, for example, $E : C :: 2 : 3$.

For $D : B :: 1 : 3$, by hypothesis.

Therefore $\underbrace{D + B} : B :: \underbrace{1 + 3} : 3$

$\qquad\qquad E \quad : B :: \quad 4 \quad : 3$

Similarly $B : \theta :: 2 : 2$, by hypothesis

Therefore $\underbrace{B + \theta} : B :: \underbrace{2 + 2} : 2$

$\qquad\qquad C \quad : B :: \quad 4 \quad : 2$

But $\qquad B \quad : E :: \quad 3 \quad : 4$

Therefore, by compounding the ratios, $C : E :: 3 : 2$. Q.E.D.

The proof is the same for all other bases, since it requires only that the proportion be found in the preceding base,and that each cell be equal to the cell before it together with the cell above it, which is everywhere the case.

Exercise 6.5. Locate the second base in Pascal's table.

(a) What two cells are contained in the second base?

(b) Carefully explain why (in the second base) the ratio of the upper cell to the lower "is the same as the number of cells the upper is to the top of the base as the lower is to the bottom of the base."

(c) Does Pascal compute the ratio of the upper to the lower or the ratio of the lower to the upper?

Exercise 6.6. Pascal claims of the Twelfth Consequence that "Although this proposition has an infinity of cases, I shall demonstrate it very briefly by supposing two lemmas. The first . . . that this proportion is found in the second base . . . . The second, that if this proposition is found in any base, it will necessarily be found in the following base." Let's continue Pascal's argument that if this proposition is found in the fourth base, then it will be found in the fifth base. Fill in the missing reasons for each step in the following argument that $R/C = 2/3$, and discuss whether the step would generalize to any base of the triangle or only at this particular location. You may use statements such as "the construction principle," "the 12th Consequence in the fourth base," or a known rule of algebra as a valid reason. For the following, "ctt(θ)" denotes "cells to the top of the base starting at cell θ," "ctb((B)" denotes "cells to the bottom of the base starting at cell B," and "ca(4th)b" denotes "cells along the 4th base.” Each of these abbreviations will be preceded by some number.

Statement	Reason
1. $\dfrac{B}{\theta} = \dfrac{2\,\mathrm{ctb}(B)}{2\,\mathrm{ctt}(\theta)}$	1.
2. $1 + \dfrac{B}{\theta} = 1 + \dfrac{2\,\mathrm{ctb}(B)}{2\,\mathrm{ctt}(\theta)}$	2.
3. $\dfrac{\theta + B}{\theta} = \dfrac{2\,\mathrm{ctt}(\theta) + 2\,\mathrm{ctb}(B)}{2\mathrm{ctt}(\theta)}$	3.
4. $\dfrac{\theta + B}{\theta} = \dfrac{4\,\mathrm{ca(4th)b}}{2\,\mathrm{ctt}(\theta)}$	4.
5. $\dfrac{C}{\theta} = \dfrac{4\,\mathrm{ca(4th)b}}{2\mathrm{ctt}(\theta)}$	5.
6. $\dfrac{\lambda}{\theta} = \dfrac{1\,\mathrm{ctt}(\lambda)}{3\,\mathrm{ctb}(\theta)}$	6.
7. $1 + \dfrac{\lambda}{\theta} = 1 + \dfrac{1\,\mathrm{ctt}(\lambda)}{3\,\mathrm{ctb}(\theta)}$	7.
8. $\dfrac{\theta + \lambda}{\theta} = \dfrac{3\,\mathrm{ctb}(\theta) + 1\,\mathrm{ctt}(\lambda)}{3\,\mathrm{ctb}(\theta)}$	8.
9. $\dfrac{\theta + \lambda}{\theta} = \dfrac{4\,\mathrm{ca(4th)b}}{3\,\mathrm{ctb}(\theta)}$	9.
10. $\dfrac{R}{\theta} = \dfrac{4\,\mathrm{ca(4th)b}}{3\,\mathrm{ctb}(\theta)}$	10.
11. $\dfrac{R}{C} = \dfrac{R}{\theta} \cdot \dfrac{\theta}{C} = \dfrac{2\,\mathrm{ctt}(\theta)}{3\,\mathrm{ctb}(\theta)}$	11.
12. $\dfrac{R}{C} = \dfrac{2\,\mathrm{ctt}(R)}{3\,\mathrm{ctb}(C)} = \dfrac{2}{3}$	12.

Exercise 6.7. Assuming that Pascal’s Twelfth Consequence holds in the fifth base, show that $F/M = 4/2$. Arrange your work in a step-by-step argument and offer a reason why each step holds, much like in Exercise (6.6).

Exercise 6.8. In this exercise we show that if Pascal’s Twelfth Consequence holds in the 20th base, then it necessarily holds in the 21st base (for a specific ratio, at least).

(a) Assuming that the Twelfth Consequence holds in the 20th base, what can be concluded about the ratios

$$E_{15,6}/E_{16,5} \quad \text{and} \quad E_{14,7}/E_{15,6}\ ?$$

(b) Consider cells $E_{15,7}$ and $E_{16,6}$ in the 21st base. In a step-by-step argument, compute $E_{15,7}/E_{16,6}$. You may use the construction principle, the Twelfth Consequence in the

20th base, and, of course, known rules of algebra. Do not use the actual values of the entries in the 20th base. Be sure to offer a reason why each step holds.

Exercise 6.9. We have seen that Pascal's Twelfth Consequence easily holds in the second base. Suppose, as Pascal claims, "that if this proposition is found in any base, it will necessarily be found in the following base." Explain how this would imply that the Twelfth Consequence holds in every base of his table, no matter how far it is continued.

Exercise 6.10. Extra. Let us now show that if the Twelfth Consequence holds in base $b-1$, where b is a natural number, ($b \geq 3$), then the Twelfth Consequence holds in the following base, b. Suppose that

$$\frac{E_{k-1,\,n+1}}{E_{k,\,n}} = \frac{k-1}{n}$$

for all natural numbers n, k with $n+k=b$. In a step-by-step argument, show that

$$\frac{E_{k-1,\,n+2}}{E_{k,\,n+1}} = \frac{k-1}{n+1}$$

Be sure to offer a reason why each step holds.

Exercise 6.11. Let's investigate how Pascal's Twelfth Consequence can be used to find algebraic formulas for the figurate numbers. Let T_n denote the nth triangular number, which is a figurate number of order 3. In our notation, $T_n = E_{3,\,n}$.

(a) Use the Twelfth Consequence to compute

$$\frac{E_{2,\,n+1}}{E_{3,\,n}} \quad \text{and} \quad \frac{E_{1,\,n+2}}{E_{2,\,n+1}}.$$

(b) Use the First Consequence to compute $E_{1,\,n+2}$.

(c) Use parts (a) and (b) to find a formula for $E_{3,\,n}$ in terms of n. You may wish to use

$$\frac{E_{3,\,n}}{E_{2,\,n+1}} \quad \text{and} \quad \frac{E_{2,\,n+1}}{E_{1,\,n+2}}.$$

(d) How could the computation of $E_{n,\,3}$ be used to find a formula for T_n as well?

Exercise 6.12. Recall that P_n denotes the nth pyramidal number, which is a figurate number of order 4. In our notation, $P_n = E_{4,\,n}$. Using the ideas of Exercise (6.11), find a formula for P_n that depends only on n. (There are several ways to implement Exercise (6.11).)

Exercise 6.13. Using the ideas of Exercise (6.11), find a formula for Q_n, the nth triangulo-triangular number. Your formula should depend only on n.

Exercise 6.14. Extra. Use Pascal's Twelfth Consequence to find a formula for $E_{k,\,n}$ that depends only on k and n. Be sure to explain your work.

Notes to the Instructor

This project is written for an entry-level general education course in undergraduate mathematics and builds on the project "Construction of the Figurate Numbers," available in the TRIUMPHS series [1]. The project covers Blaise Pascal's (1623–1662) arrangement of the figurate numbers (binomial coefficients) into one table, known as Pascal's triangle [7]. The project continues with several patterns among the figurate numbers identified by Pascal that then become apparent. Going well beyond pattern recognition, Pascal wishes to justify why these patterns persist no matter how far this table is continued. In doing so, he develops the idea of mathematical induction, stated verbally. The main mathematical topics of this project are then computation of binomial coefficients (as figurate numbers) and mathematical induction, with little algebraic formalism.

Students should have a good knowledge of arithmetic and facility with fractions, since these appear when finding quotients of figurate numbers. Subscript notation is used throughout the project, such as P_1, P_2, P_3, for the first, second and third pyramidal numbers (defined in the project). A generic subscript appears as P_n. In the final section, Pascal's Twelfth Consequence, double subscripts, $E_{k,n}$, are used to denote the entry in row k and column n of Pascal's table (triangle). Only Pascal's First, Second, Fifth and Twelfth Consequences are covered in this project. Instructors wishing to explore additional patterns or seeking independent study topics for students should consult Pascal's treatise [7]. Applications of Pascal's triangle to the binomial theorem or the combinatorics of choosing k many objects from m are also not covered in the project, but do appear in Pascal's work. For a more algebraic treatment of this material, see David Pengelley's project "Treatise on the Arithmetical Triangle" [8], written for a course in discrete mathematics for math majors.

To use this project, it is not necessary to have completed the prequel "Construction of the Figurate Numbers," although the instructor should read that project for background material. The present project begins with a section "Review of the Figurate Numbers," which should be covered in any event, since this section introduces Pascal's term "order" of a figurate number, referring to the row number in which a figurate number appears in his table. The instructor should carefully read the excerpts from Pascal's paper and be prepared to discuss them with the class. The first few exercises in each section are fairly transparent and then build in sophistication. Exercises requiring mostly algebra are marked as "Extra," and may be counted as extra credit at the instructor's discretion. The instructor should work any exercise before assignment.

It is suggested that the sections on the First, Second, Fifth and Twelfth Consequences be worked in order to appreciate Pascal's development of mathematical induction, although the verbal statement of induction appears only in the Twelfth Consequence. The project in its entirety requires about five weeks in a general education course. If time remains, those seeking applications of Pascal's triangle could cover the binomial theorem or the combinatorics of choosing k-many objects from m, which are treated in his original paper [7].

When working through Pascal's paper, the instructor may wish to hand out a separate copy of Pascal's Arithmetical Triangle for ease of reference. This appears on a following page, which can be easily copied and distributed to the class. The excerpt from Pascal's treatise in the project contains a copy of this table with French labels for the "parallel rows" and "perpendicular rows."

LaTeX code of this entire PSP is available from the author by request. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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For more information about TRIUMPHS, visit
http://webpages.ursinus.edu/nscoville/TRIUMPHS.html.

Z	1	2	3	4	5	6	7	L 8	9	10
1	G 1	σ 1	π 1	λ 1	μ 1	δ 1	ζ 1	1	1	1
2	φ 1	ψ 2	θ 3	R 4	S 5	N 6	7	8	9	
3	A 1	B 3	C 6	ω 10	ξ 15	21	28	36		
4	D 1	E 4	F 10	ρ 20	Y 35	56	84			
5	H 1	M 5	K 15	35	70	126				
6	P 1	Q 6	21	56	126					
7	V 1	7	28	84						
T 8	1	8	36							
9	1	9								
10	1									

Pascal's Arithmetical Triangle.

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